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A SEMI-LAGRANGIAN DISCONTINUOUS GALERKIN SUPERCONVERGENCE

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Abstract. We show a superconvergence property for the Semi-Lagrangian Discontinuous Galerkin scheme of arbitrary degree in the case of constant linear advection equation with periodic boundary conditions.

1. Introduction

The aim of this paper is to show a superconvergence property for the Semi-Lagrangian Discontinuous Galerkin scheme (SLDG) of arbitrary degree. Such a scheme has been developped already in [7] and then more recently in [9, 11, 5] for Vlasov-Maxwell/Poisson applications. One key point, in such applications, is to use directional splitting which leads to a succession of constant advection problems and the scheme has the advantage of not being restricted to a CFL condition. The case of non constant advection is more delicate and can lead to different strategies for the evaluation of the flux, which has to be further approximated; see [9] for a discussion, and [10] for a pionnering work on the subject in a general setting. The SLDG scheme continues to be under investigations, see [1, 2, 8].

Superconvergence of the Discontinuous Galerkin method has been the subject of several developments. In [6] a Fourier approach is used to analyze the superconvergence properties. While being limited to uniform mesh with periodic boundary conditions and linear problems, Fourier approach permits to give precise information of the error. It is however often restricted to low degrees as symbolic computations become more and more complex when the degree increases. Note that other technics have been developed for treating more general cases, as the post-processing introduced in [4]; see also many other references in [6].

We consider here the superconvergence of the SLDG scheme in the case of constant linear advection equation with periodic boundary conditions. We are able to give a result for an *arbitrary* degree, which seems not to have been considered, to our knowledge, for the SLDG scheme. The technic of proof is also new and may have its own interest. It uses a *vectorial* Fourier decomposition, Cauchy-Schwarz inequality and the Euler-MacLaurin formula.

The paper is organized as follows. In Section 2, we introduce the Semi-Lagrangian Discontinuous Galerkin scheme. The statement of the theorem of superconvergence is given in Section 3. In Section 4, we describe the structure of the truncation and numerical errors. An eigenstructure analysis of the scheme's amplification matrix is proceeded in Section 5. In Section 6, we resume the previous results and conclude

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the proof. Numerical results showing superconvergence are presented in Section 7 and conclusion is given in Section 8.

2. Semi-Lagrangian Discontinuous Galerkin scheme

2.1. **Notations.** The equation under study is the constant linear advection equation

$$\begin{aligned} \partial_t f + a\partial_x f &= 0 \qquad\qquad (x,t)\in[0,1]\times[0,+\infty[\\ f(0,x) &= f_0(x) \qquad\qquad x\in[0,1] \end{aligned}$$

with a constant velocity $a > 0$.

Let $\Omega = [0,1]$ be the domain, which is divided in N cells :

$$C_i = [x_{i-1/2}, x_{i+1/2}], \quad i = 0, ..., N-1.$$

We suppose here that the mesh is uniform: the space step Δx satisfies

$$\Delta x = x_{i+1/2} - x_{i-1/2} = \frac{1}{N}, \qquad i = 0, ..., N-1.$$

We define also the time step Δt which is also suppose to be constant and write $t^n = n\Delta t$. Periodic boundary conditions will be used.

2.2. **SLDG scheme.** Let $d \in \mathbb{N}$. On each cell $C_i = [x_{i-1/2}, x_{i+1/2}]$, we put $d+1$ Gauss points denoted by $\{x_{ij}\}_{(i,j)\in\{0,...,N-1\}\times\{0,...,d\}}$. Denoting by $\{\alpha_j\}_{j\in\{0,...,d\}}$ the Gauss points in the interval $[0,1]$ and $\{\omega_j\}_{j\in\{0,...,d\}}$ their associated weights, we first introduce the Lagrange polynomials at points α_j restricted to the interval $[0,1]$:

$$\varphi_j(x) = \prod_{\ell,\ell\neq j} \frac{x-\alpha_\ell}{\alpha_j-\alpha_\ell} \text{ for } x\in[0,1], \qquad \varphi_j(x) = 0 \text{ otherwise}$$

and the corresponding polynomial for the cell C_i :

$$\varphi_{ij}(x) = \varphi_j\left(\frac{x - x_{i-1/2}}{\Delta x}\right).$$

By writing $f^n \approx f(t^n,\cdot)$ in the form $f^n(x) = \sum_{i,j} f_{ij}^n \varphi_{ij}(x)$, the degrees of freedom $f_{ij}^n \approx f(t^n, \alpha_{ij})$ are given by

$$\omega_j \Delta x f_{ij}^n = \int_{\mathbb{R}} f^n(x)\varphi_{ij}(x)dx.$$

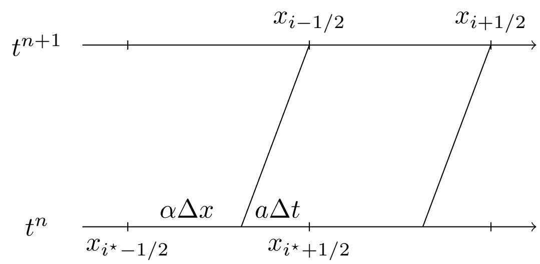
Using the advection equation for updating the degrees of freedom, we get the semi-Lagrangian discontinuous Galerkin (SLDG) scheme:

$$\omega_j \Delta x f_{ij}^{n+1} = \int_{\mathbb{R}} f^n(x - a\Delta t)\varphi_{ij}(x)dx.$$

This leads to

$$\omega_j \Delta x f_{ij}^{n+1} = \sum_{k=0}^{N-1}\sum_{\ell=0}^{d} f_{k\ell}^n \int_{\mathbb{R}} \varphi_\ell\left(\frac{x - a\Delta t - x_{k-1/2}}{\Delta x}\right)\varphi_j\left(\frac{x - x_{i-1/2}}{\Delta x}\right)dx.$$

By defining $i^\star$ and α such that $x_{i-1/2} - a\Delta t = x_{i^\star-1/2} + \alpha\Delta x$,



and using the change of variable $x = x_{i-1/2} + s\Delta x$, we get :

$$\omega_j \Delta x f_{ij}^{n+1} = \Delta x \sum_{k=0}^{N-1} \sum_{\ell=0}^{d} f_{k\ell}^n \int_{\mathbb{R}} \varphi_\ell(i^\star - k + \alpha + s)\varphi_j(s)ds$$

and finally, obtain the following explicit formula for the SLDG scheme:

$$\omega_j f_{ij}^{n+1} = \sum_{\ell=0}^{d} f_{i^\star,\ell}^n \int_{\mathbb{R}} \varphi_\ell(\alpha+s)\varphi_j(s)ds + \sum_{\ell=0}^{d} f_{i^\star+1,\ell}^n \int_{\mathbb{R}} \varphi_\ell(\alpha+s-1)\varphi_j(s)ds. \tag{2.1}$$

We define the L^2 discrete norm as follows:

$$\|z\|_2^2 = \frac{1}{N}\sum_{i=0}^{N-1}\sum_{j=0}^{d} \omega_j z_{i,j}^2, \;\; z = (z_{i,j}) \in \mathbb{R}^{N(d+1)}.$$

3. Superconvergence property

The goal of this paper is to prove the following theorem.

Theorem 3.1. *Consider the constant linear advection equation*

$$\begin{aligned} &\partial_t f + a\partial_x f = 0 && (x,t) \in [0,1]\times[0,T] \\ &f(0,x) = f^0(x) && x \in [0,1] \end{aligned}$$

with $f^0 \in \mathcal{C}^{2d+2}([0,1])$ and the Semi-Lagrangian Discontinuous Galerkin scheme for the discretisation of this equation. We write $-\frac{a\Delta t}{\Delta x} = i_0^\star + \alpha$ where $i_0^\star \in \mathbb{Z}$ and $0 \leq \alpha < 1$. Then there exists constants $C_1, C_2 > 0$ which depend on α, on the regularity of the solution and independent of time T such that the numerical error at point x_{ij} : $e_{ij}^n = f_{ij}^n - f(t^n, x_{ij})$ is bounded in the L^2 discrete norm:

$$\|\boldsymbol{e}^n\|_2 \leq C_1 \Delta x^{d+1} + nC_2\Delta x^{2d+2}.$$

Remark 3.2. We can restrict us to the case where $0 < \frac{a\Delta t}{\Delta x} < 1$ by using the fact that the scheme is exact when $\frac{a\Delta t}{\Delta x} \in \mathbb{Z}$.

Remark 3.3. One advantage of this new estimation is to have convergence over a *longer* time. In fact, if $1 \leq \beta \leq d$ and if we fix the values of Δx and Δt, we are looking for the largest time $T = n\Delta t$ such that

$$\|\mathbf{e}^n\|_2 \leq C_4 \Delta x^\beta$$

where C_4 is a constant.

(1) In the classical case, we have $\|\mathbf{e}^n\|_2 \leq C_5 n \Delta x^{d+1}$ which leads to

$$T \leq C_6 \Delta x^{\beta - d}$$

where C_5, C_6 are constants.

(2) In the superconvergence case, we get

$$T \leq C_7 \Delta x^{\beta - 2d - 1}$$

where C_7 is a constant.

Remark 3.4. The scheme considered here is different from that described in [6]. The scheme in [6] corresponds to the exponential integrator when $\alpha \to 0$ of the scheme considered in this paper, which is here not covered by the analysis, but may be adapted. See [3] for a similar example.

Remark 3.5. The degree is here arbitrary. Fourier approach, as in [6], is possible for low degrees (see Section 7). It gives more precise informations but the computational complexity increases strongly with the degree, and seems to become impractible for higher degrees.

4. Truncation and numerical errors

4.1. **Truncation error.** We write $\mathbf{x} = (x_{ij})_{i=0..N-1, j=0..d}$.

Notation 4.1. $\mathcal{S} : \mathbb{R}^{N(d+1)} \to \mathbb{R}^{N(d+1)}$ is the Semi-Lagrangian Discontinuous Galerkin scheme operator given by (2.1).

Notation 4.2. The truncation error at point x_{ij} and time t^n is defined by

$$g_{ij}^n = \frac{1}{\Delta t}\left(f(t^{n+1}, x_{ij}) - \mathcal{S}(f(t^n, \mathbf{x}))_{ij} \right).$$

The following proposition gives an expression of the truncation error.

Proposition 4.3. There exists constants E_j^k independent of i and n and dependent on α such that

$$g_{ij}^n = \sum_{k=0}^{2d+1} E_j^k \frac{\Delta x^k}{\Delta t} \partial_x^k f(t^n, x_{ij}) + \mathcal{O}\left(\frac{\Delta x^{2d+2}}{\Delta t} \right).$$

Proof. We assume that $0 < \frac{a\Delta t}{\Delta x} < 1$, thus $i^\star = i - 1$ and $\alpha = 1 - \frac{a \Delta t}{\Delta x}$. The computation of the truncation error at point x_{ij} defined by

$$\frac{1}{\Delta t}\left(f(t^{n+1}, x_{ij}) - \mathcal{S}(f(t^n, \mathbf{x}))_{ij} \right)$$

gives :

$$\frac{1}{\Delta t}\Bigg(f(t^n + \Delta t, x_{ij}) - \frac{1}{\omega_j} \sum_{j'=0}^{d} f(t^n, (i - 1 + \alpha_{j'})\Delta x) \int_{s=\alpha}^{1} \varphi_{j'}(s) \varphi_j(s - \alpha) ds$$
$$- \frac{1}{\omega_j} \sum_{j'=0}^{d} f(t^n, (i + \alpha_{j'})\Delta x) \int_{s=0}^{\alpha} \varphi_{j'}(s) \varphi_j(s + 1 - \alpha) ds \Bigg).$$

We use the constant advection equation to go back :

$$f(t^n + \Delta t, x_{ij}) = f(t^n, x_{ij} - a\Delta t)$$

and then, by Taylor expansion at point x_{ij}, the truncation error at point x_{ij} reads

$$\sum_{k=0}^{2d+1} E_j^k \frac{\Delta x^k}{\Delta t} \partial_x^k f(t^n, x_{ij}) + \mathcal{O}\left(\frac{\Delta x^{2d+2}}{\Delta t}\right)$$

where

$$\begin{aligned} E_j^k &= \frac{(\alpha-1)^k}{k!} - \frac{1}{\omega_j} \sum_{j'=0}^{d} \left(\frac{(\alpha_{j'}-\alpha_j-1)^k}{k!} \int_{s=\alpha}^{1} \varphi_{j'}(s)\varphi_j(s-\alpha)ds + \right. \\ & \left. \frac{(\alpha_{j'}-\alpha_j)^k}{k!} \int_{s=0}^{\alpha} \varphi_{j'}(s)\varphi_j(s+1-\alpha)ds \right). \end{aligned}$$

□

We first can state that the first coefficients are zero, which leads to classical convergence estimates of order $d+1$:

Proposition 4.4. For all $0 \le k \le d$ and all $0 \le j \le d$, the coefficients of the truncation error, defined in Proposition 4.3, satisfy

$$E_j^k = 0.$$

Proof. We begin with the case $k = 0$:

$$\begin{aligned} E_j^0 &= 1 - \frac{1}{\omega_j} \sum_{j'=0}^{d} \left(\int_{s=\alpha}^{1} \varphi_{j'}(s)\varphi_j(s-\alpha)ds \; + \; \int_{s=0}^{\alpha} \varphi_{j'}(s)\varphi_j(s+1-\alpha)ds \right) \\ &= 1 - \frac{1}{\omega_j} \left[\int_{s=\alpha}^{1} \left(\sum_{j'=0}^{d} \varphi_{j'}(s) \right) \varphi_j(s-\alpha)ds + \int_{s=0}^{\alpha} \left(\sum_{j'=0}^{d} \varphi_{j'}(s) \right) \varphi_j(s+1-\alpha)ds \right]. \end{aligned}$$

We have

$$1 - \sum_{j'=0}^{d} \varphi_{j'}(s) \equiv 0$$

since the left term is a polynomial of degree d with $d+1$ zeros $(\alpha_0, ..., \alpha_d)$. Then, we get

$$\begin{aligned} E_j^0 &= 1 - \frac{1}{\omega_j} \left(\int_{s=0}^{1-\alpha} \varphi_j(s)ds + \int_{s=1-\alpha}^{1} \varphi_j(s)ds \right) \\ &= 1 - \frac{1}{\omega_j} \int_{s=0}^{1} \varphi_j(s)ds \\ &= 1 - \sum_{i=0}^{d} \varphi_j(\alpha_i) \\ &= 1 - \sum_{i=0}^{d} \delta_{ij} \\ &= 0. \end{aligned}$$

The coefficient E_j^k for $1 \leq k \leq d$ reads :

$$\begin{aligned} E_j^k &= \frac{(\alpha-1)^k}{k!} - \frac{1}{\omega_j}\sum_{j'=0}^{d}\left(\frac{(\alpha_{j'}-\alpha_j-1)^k}{k!}\int_{s=\alpha}^{1}\varphi_{j'}(s)\varphi_j(s-\alpha)ds + \right. \\ &\quad \left. \frac{(\alpha_{j'}-\alpha_j)^k}{k!}\int_{s=0}^{\alpha}\varphi_{j'}(s)\varphi_j(s+1-\alpha)ds\right). \end{aligned}$$

For $1 \leq k \leq d$,

$$\sum_{j'=0}^{d}(\alpha_{j'}-\alpha_j-1)^k\varphi_{j'}(s) \equiv (s-\alpha_j-1)^k$$

and

$$\sum_{j'=0}^{d}(\alpha_{j'}-\alpha_j)^k\varphi_{j'}(s) \equiv (s-\alpha_j)^k$$

because the left and right terms are polynomials of degree at most d with $d+1$ commune values at α_i ($i = 0, \ldots, d$). Then :

$$\begin{aligned} E_j^k &= \frac{1}{k!}\left[(\alpha-1)^k - \frac{1}{\omega_j}\int_{s=\alpha}^{1}(s-\alpha_j-1)^k\varphi_j(s-\alpha)ds + \right. \\ &\quad \left. -\frac{1}{\omega_j}\int_{s=0}^{\alpha}(s-\alpha_j)^k\varphi_j(s+1-\alpha)ds\right]. \end{aligned}$$

By change of variable, we obtain

$$E_j^k = \frac{1}{k!}\left[(\alpha-1)^k - \frac{1}{\omega_j}\int_{s=0}^{1}(t-1+\alpha-\alpha_j)^k\varphi_j(t)dt\right].$$

The polynomial

$$t \mapsto (t-1+\alpha-\alpha_j)^k\varphi_j(t)$$

is of degree less or equal than $2d$, therefore we get by Gauss quadrature formula :

$$E_j^k = \frac{1}{k!}\left[(\alpha-1)^k - (\alpha_j-1+\alpha-\alpha_j)^k\right] = 0.$$

$\square$

In order to obtain a superconvergence property, we have the following weaker property which is valid also for higher order terms, until the degree $2d+1$:

Proposition 4.5. For all $k = 0, \ldots, 2d+1$, the coefficients of the truncation error, defined in Proposition 4.3, satisfy

$$\sum_{j=0}^{d}\omega_j E_j^k = 0.$$

Proof. Note that the cases $k = 0, \dots, d$ are already given by the previous proposition. The coefficient E_j^k reads

$$E_j^k = \frac{(\alpha-1)^k}{k!} - \frac{1}{\omega_j}\sum_{j'=0}^{d}\left(\frac{(\alpha_{j'}-\alpha_j-1)^k}{k!}\int_{s=\alpha}^{1}\varphi_{j'}(s)\varphi_j(s-\alpha)ds + \frac{(\alpha_{j'}-\alpha_j)^k}{k!}\int_{s=0}^{\alpha}\varphi_{j'}(s)\varphi_j(s+1-\alpha)ds\right)$$

thus

$$\sum_{j=0}^{d}\omega_j E_j^k = \frac{1}{k!}\left((\alpha-1)^k - \sum_{j=0}^{d}\sum_{j'=0}^{d}\left((\alpha_{j'}-\alpha_j-1)^k\int_{s=\alpha}^{1}\varphi_{j'}(s)\varphi_j(s-\alpha)ds + (\alpha_{j'}-\alpha_j)^k\int_{s=0}^{\alpha}\varphi_{j'}(s)\varphi_j(s+1-\alpha)ds\right)\right).$$

We thus have to establish that for all $d+1 \le k \le 2d+1$:

$$\sum_{j=0}^{d}\sum_{j'=0}^{d}\left((\alpha_{j'}-\alpha_j-1)^k\int_{s=\alpha}^{1}\varphi_{j'}(s)\varphi_j(s-\alpha)ds+(\alpha_{j'}-\alpha_j)^k\int_{s=0}^{\alpha}\varphi_{j'}(s)\varphi_j(s+1-\alpha)ds\right) = (\alpha-1)^k.$$

In this proof, we use the following properties :

(P1) For all polynomial P of degree less or equal than d, we have

$$\sum_{j=0}^{d}P(\alpha_j)\varphi_j(s) \equiv P(s).$$

(P2) For all polynomial P of degree less or equal than $2d+1$, we have

$$\int_0^1 P(x)dx = \sum_{i=0}^{d}\omega_i P(\alpha_i).$$

By using the binomial theorem and separating the cases $r \le d$ and $r > d$, we get :

$$\begin{aligned}
A &:= \sum_{j=0}^{d}\sum_{j'=0}^{d}\left((\alpha_{j'}-\alpha_j-1)^k\int_{s=\alpha}^{1}\varphi_{j'}(s)\varphi_j(s-\alpha)ds + (\alpha_{j'}-\alpha_j)^k\int_{s=0}^{\alpha}\varphi_{j'}(s)\varphi_j(s+1-\alpha)ds\right)\\
&= \sum_{j,j'=0}^{d}\sum_{r=0}^{d}\int_{s=\alpha}^{1}\binom{k}{r}\alpha_{j'}^r(-\alpha_j-1)^{k-r}\varphi_{j'}(s)\varphi_j(s-\alpha)ds\\
&\quad+\sum_{j,j'=0}^{d}\sum_{r=d+1}^{k}\int_{s=\alpha}^{1}\binom{k}{r}\alpha_{j'}^r(-\alpha_j-1)^{k-r}\varphi_{j'}(s)\varphi_j(s-\alpha)ds\\
&\quad+\sum_{j,j'=0}^{d}\sum_{r=0}^{d}\int_{s=0}^{\alpha}\binom{k}{r}\alpha_{j'}^r(-\alpha_j)^{k-r}\varphi_{j'}(s)\varphi_j(s+1-\alpha)ds\\
&\quad+\sum_{j,j'=0}^{d}\sum_{r=d+1}^{k}\int_{s=0}^{\alpha}\binom{k}{r}\alpha_{j'}^r(-\alpha_j)^{k-r}\varphi_{j'}(s)\varphi_j(s+1-\alpha)ds.
\end{aligned}$$

As $k \leq 2d+1$, the relation $r > d$ implies $k-r \leq d$. We can use the property (P1) in each term :

$$\begin{aligned} A &= \sum_{j=0}^{d}\sum_{r=0}^{d}\int_{s=\alpha}^{1}\binom{k}{r}s^r(-\alpha_j-1)^{k-r}\varphi_j(s-\alpha)ds \\ &\quad+\sum_{j'=0}^{d}\sum_{r=d+1}^{k}\int_{s=\alpha}^{1}\binom{k}{r}\alpha_{j'}^r(-s+\alpha-1)^{k-r}\varphi_{j'}(s)ds \\ &\quad+\sum_{j=0}^{d}\sum_{r=0}^{d}\int_{s=0}^{\alpha}\binom{k}{r}s^r(-\alpha_j)^{k-r}\varphi_j(s+1-\alpha)ds \\ &\quad+\sum_{j'=0}^{d}\sum_{r=d+1}^{k}\int_{s=0}^{\alpha}\binom{k}{r}\alpha_{j'}^r(-s+\alpha-1)^{k-r}\varphi_{j'}(s)ds \\ &=: (1)+(2)+(3)+(4). \end{aligned}$$

We first calculate the sum (2) + (4) since only the limits of integration differ in these two expressions.

$$(2)+(4) = \sum_{j'=0}^{d}\sum_{r=d+1}^{k}\int_{s=0}^{1}\binom{k}{r}\alpha_{j'}^r(-s+\alpha-1)^{k-r}\varphi_{j'}(s)ds.$$

As $s \mapsto (-s+\alpha-1)^{k-r}\varphi_{j'}(s)$ is a polynomial of degree less or equal than $2d$, we can apply the property (P2) :

$$(2)+(4) = \sum_{j'=0}^{d}\sum_{r=d+1}^{k}\binom{k}{r}\alpha_{j'}^r(-\alpha_{j'}+\alpha-1)^{k-r}\omega_{j'}.$$

Applying property (P2), this time to the polynomial $s \mapsto s^r(-s+\alpha-1)^{k-r}$ which is of degree $2d+1$:

$$(2)+(4) = \sum_{r=d+1}^{k}\binom{k}{r}\int_0^1 s^r(-s+\alpha-1)^{k-r}\varphi_{j'}(s)ds.$$

Turning to the computation of (1)+(3). By making the change of variables $s = s-\alpha$ and $s = s+1-\alpha$, we get

$$\begin{aligned} (1)+(3) &= \sum_{j=0}^{d}\sum_{r=0}^{d}\int_{s=0}^{1-\alpha}\binom{k}{r}(s+\alpha)^r(-\alpha_j-1)^{k-r}\varphi_j(s)ds \\ &\quad+\sum_{j=0}^{d}\sum_{r=0}^{d}\int_{s=1-\alpha}^{1}\binom{k}{r}(s-1+\alpha)^r(-\alpha_j)^{k-r}\varphi_j(s)ds. \end{aligned}$$

Separating again the case $r \leq d$ et $r > d$:

$$\begin{aligned}
(1)+(3) \quad = \quad & \sum_{j=0}^{d}\sum_{r=0}^{k}\int_{s=0}^{1-\alpha}\binom{k}{r}(s+\alpha)^r(-\alpha_j-1)^{k-r}\varphi_j(s)ds \\
& -\sum_{j=0}^{d}\sum_{r=d+1}^{k}\int_{s=0}^{1-\alpha}\binom{k}{r}(s+\alpha)^r(-\alpha_j-1)^{k-r}\varphi_j(s)ds \\
& +\sum_{j=0}^{d}\sum_{r=0}^{k}\int_{s=1-\alpha}^{1}\binom{k}{r}(s-1+\alpha)^r(-\alpha_j)^{k-r}\varphi_j(s)ds \\
& -\sum_{j=0}^{d}\sum_{r=d+1}^{k}\int_{s=1-\alpha}^{1}\binom{k}{r}(s-1+\alpha)^r(-\alpha_j)^{k-r}\varphi_j(s)ds.
\end{aligned}$$

We can then use the binomial theorem and the property (P1) for polynomials $(-s-1)^{k-r}$ and $(-s)^{k-r}$:

$$\begin{aligned}
(1)+(3) \quad = \quad & \sum_{j=0}^{d}\int_{s=0}^{1-\alpha}(s+\alpha-\alpha_j-1)^k\varphi_j(s)ds \\
& -\sum_{j=0}^{d}\sum_{r=d+1}^{k}\int_{s=0}^{1-\alpha}\binom{k}{r}(s+\alpha)^r(-\alpha_j-1)^{k-r}\varphi_j(s)ds \\
& +\sum_{j=0}^{d}\int_{s=1-\alpha}^{1}(s-1+\alpha-\alpha_j)^k\varphi_j(s)ds \\
& -\sum_{j=0}^{d}\sum_{r=d+1}^{k}\int_{s=1-\alpha}^{1}\binom{k}{r}(s-1+\alpha)^r(-\alpha_j)^{k-r}\varphi_j(s)ds \\
= \quad & \sum_{j=0}^{d}\int_{s=0}^{1}(s+\alpha-\alpha_j-1)^k\varphi_j(s)ds \\
& -\sum_{r=d+1}^{k}\int_{s=0}^{1-\alpha}\binom{k}{r}(s+\alpha)^r(-s-1)^{k-r}ds \\
& -\sum_{r=d+1}^{k}\int_{s=1-\alpha}^{1}\binom{k}{r}(s-1+\alpha)^r(-s)^{k-r}ds.
\end{aligned}$$

We make the change of variable $s = s - 1$ in the last term :

$$\begin{aligned}
(1) + (3) \quad &= \quad \sum_{j=0}^{d} \int_{s=0}^{1} (s + \alpha - \alpha_j - 1)^k \varphi_j(s) ds \\
&\quad - \sum_{r=d+1}^{k} \int_{s=0}^{1-\alpha} \binom{k}{r} (s + \alpha)^r (-s - 1)^{k-r} ds \\
&\quad - \sum_{r=d+1}^{k} \int_{s=-\alpha}^{0} \binom{k}{r} (s + \alpha)^r (-s - 1)^{k-r} ds \\
&= \quad \sum_{j=0}^{d} \int_{s=0}^{1} (s + \alpha - \alpha_j - 1)^k \varphi_j(s) ds \\
&\quad - \sum_{r=d+1}^{k} \int_{s=-\alpha}^{1-\alpha} \binom{k}{r} (s + \alpha)^r (-s - 1)^{k-r} ds.
\end{aligned}$$

The change of variable $s = s + \alpha$ in the last term follows to :

$$\begin{aligned}
(1) + (3) \quad &= \quad \sum_{j=0}^{d} \int_{s=0}^{1} (s + \alpha - \alpha_j - 1)^k \varphi_j(s) ds \\
&\quad - \sum_{r=d+1}^{k} \int_{s=0}^{1} \binom{k}{r} s^r (-s + \alpha - 1)^{k-r} ds.
\end{aligned}$$

We sum finally our two intermediate results and we obtain:

$$\begin{aligned}
(1) + (2) + (3) + (4) \quad &= \quad \sum_{j=0}^{d} \int_{s=0}^{1} (s + \alpha - \alpha_j - 1)^k \varphi_j(s) ds \\
&= \quad \sum_{j=0}^{d} \sum_{r=0}^{k} \binom{k}{r} \int_{s=0}^{1} (s + \alpha)^r (-\alpha_j - 1)^{k-r} \varphi_j(s) ds.
\end{aligned}$$

We separate the cases $r \leq d$ et $r > d$:

$$\begin{aligned}
(1) + (2) + (3) + (4) \quad &= \quad \sum_{j=0}^{d} \sum_{r=0}^{d} \binom{k}{r} \int_{s=0}^{1} (s + \alpha)^r (-\alpha_j - 1)^{k-r} \varphi_j(s) ds \\
&\quad + \sum_{j=0}^{d} \sum_{r=d+1}^{k} \binom{k}{r} \int_{s=0}^{1} (s + \alpha)^r (-\alpha_j - 1)^{k-r} \varphi_j(s) ds.
\end{aligned}$$

In the first term, we use the property (P2) with the polynomial $s \mapsto (s + \alpha)^r \varphi_j(s)$ of maximal degree $2d$. In the second term, we use the property (P1) with the polynomial $s \mapsto (-s-1)^{k-r}$ of maximal degree d :

$$\begin{aligned}(1)+(2)+(3)+(4) &= \sum_{j=0}^{d}\sum_{r=0}^{d}\binom{k}{r}\omega_j(\alpha_j+\alpha)^r(-\alpha_j-1)^{k-r} \\ &\quad + \sum_{r=d+1}^{k}\binom{k}{r}\int_{s=0}^{1}(s+\alpha)^r(-s-1)^{k-r}ds.\end{aligned}$$

As the polynomial $s \mapsto (s+\alpha)^r(-s-1)^{k-r}$ is of degree $k \le 2d+1$, we can apply (P2) in the first term :

$$\begin{aligned}(1)+(2)+(3)+(4) &= \sum_{r=0}^{d}\binom{k}{r}\int_{s=0}^{1}(s+\alpha)^r(-s-1)^{k-r}ds \\ &\quad + \sum_{r=d+1}^{k}\binom{k}{r}\int_{s=0}^{1}(s+\alpha)^r(-s-1)^{k-r}ds \\ &= \int_{s=0}^{1}\sum_{r=0}^{k}\binom{k}{r}(s+\alpha)^r(-s-1)^{k-r}ds \\ &= (\alpha-1)^k.\end{aligned}$$

which completes the proof. $\square$

4.2. **Numerical error.** The numerical error at point x_{ij} and time t^n is defined by

$$e_{ij}^n = f(t^n, x_{ij}) - \mathcal{S}^n(f(t^0, \mathbf{x}))_{ij}$$

We classically relate the numerical error with the truncation error:

Proposition 4.6. We have

$$\mathbf{e}^n = \mathcal{S}\mathbf{e}^{n-1} + \Delta t \mathbf{g}^{n-1}.$$

By iteration, we get

$$\mathbf{e}^n = \Delta t \sum_{\ell=0}^{n-1}\mathcal{S}^\ell \mathbf{g}^{n-1-\ell}. \tag{4.1}$$

Proof. By definition, we have

$$\mathbf{e}^n = f(t^n, \mathbf{x}) - \mathcal{S}(\mathcal{S}^{n-1}(f(t^0, \mathbf{x})))$$

and

$$\mathcal{S}^{n-1}(f(t^0, \mathbf{x})) = f(t^{n-1}, \mathbf{x}) - \mathbf{e}^{n-1}.$$

This leads to

$$\mathbf{e}^n = \mathcal{S}\mathbf{e}^{n-1} + f(t^n, \mathbf{x}) - \mathcal{S}(f(t^{n-1}, \mathbf{x})),$$

which gives the result. $\square$

5. Eigenstructure analysis

We see in the expression of the numerical error (4.1) that we have to deal with powers of $\mathcal{S}$. Such a study can be tackled by looking at the spectral decomposition of $\mathcal{S}$.

For each cell $i = 0, \dots, N-1$, the scheme given by (2.1) can be written as

$$\begin{pmatrix} f_{i,0}^{n+1} \\ \vdots \\ f_{i,d}^{n+1} \end{pmatrix} = A_{-1} \begin{pmatrix} f_{i-1,0}^{n} \\ \vdots \\ f_{i-1,d}^{n} \end{pmatrix} + A_0 \begin{pmatrix} f_{i,0}^{n} \\ \vdots \\ f_{i,d}^{n} \end{pmatrix}$$

where $A_{-1}, A_0 \in \mathcal{M}_{d+1}(\mathbb{R})$ are the matrices :

$$(A_{-1})_{ij} = \int_{\mathbb{R}} \varphi_j(\alpha + s)\varphi_i(s)ds = \int_{s=\alpha}^{1} \varphi_i(s-\alpha)\varphi_j(s)ds$$
$$(A_0)_{ij} = \int_{\mathbb{R}} \varphi_j(\alpha + s - 1)\varphi_i(s)ds = \int_{s=0}^{\alpha} \varphi_i(s-\alpha+1)\varphi_j(s)ds.$$

Then, in the natural basis associated to $\mathbf{x}$, the matrix of $\mathcal{S}$ is given by

$$\mathcal{S} = \begin{pmatrix} A_0 & & & A_{-1} \\ A_{-1} & A_0 & & \\ & \ddots & \ddots & \\ & & A_{-1} & A_0 \end{pmatrix} \in \mathcal{M}_{N(d+1)}(\mathbb{R})$$

The matrix $\mathcal{S}$ is a block-circulant matrix, we can then perform a vectorial Fourier decomposition. Such decomposition is already used in [5].

Proposition 5.1. We have the decomposition

$$\mathcal{S} = UDU^\star$$

where

$$D = \begin{pmatrix} D_0 & & \\ & \ddots & \\ & & D_{N-1} \end{pmatrix} \qquad U = \begin{pmatrix} U_{0,0} & \dots & U_{0,N-1} \\ \vdots & \ddots & \vdots \\ U_{N-1,0} & \dots & U_{N-1,N-1} \end{pmatrix}$$

with

$$D_m = A_0 + A_{-1} e^{\frac{2i\pi m}{N}} \qquad U_{k,\ell} = \frac{1}{\sqrt{N}} e^{\frac{2i\pi k\ell}{N}} I_{d+1}$$

and I_{d+1} is the identity matrix of size $(d+1) \times (d+1)$.

Proof. The (k, ℓ)-block of the matrix $UDU^\star$ is equal to

$$\frac{1}{N} \sum_{m=0}^{N-1} D_m e^{\frac{2i\pi m(k-\ell)}{N}} = A_0 \delta_{k\ell} + A_{-1}\delta_{k+1,\ell}$$

where δ is the Kronecker symbol. □

We are now looking for the eigenstructure of $\mathcal{S}$. For this, we are reduced to look at the eigenstructure of the matrices $D_k, k = 0, \dots, N-1$. We first consider the matrice D_0 :

Proposition 5.2. The only eigenvalue of module 1 of the matrix $D_0 := A_{-1} + A_0$ is 1, which is not multiple. Other eigenvalues have modulus strictly less than 1. Moreover, we have

$$D_0^n \underset{n\to+\infty}{\longrightarrow} G$$

where G denotes the Gauss weights matrix :

$$G = \begin{pmatrix} \omega_0 & \omega_1 & \dots & \omega_d \\ \omega_0 & \omega_1 & \dots & \omega_d \\ \vdots & \vdots & \vdots & \vdots \\ \omega_0 & \omega_1 & \dots & \omega_d \end{pmatrix}.$$

Proof. In this proof, we will use the following relations :

$$\sum_{j=0}^{d} \varphi_j(x) = 1 \qquad \text{for all } 0 \le x \le 1 \tag{5.1}$$

$$\int_0^1 \varphi_j(s)ds = \omega_j. \tag{5.2}$$

Let λ be an eigenvalue of the matrix D_0. There exists $x = (x_0, ..., x_d)$ such that $D_0 x = \lambda x$:

$$\lambda\omega_j x_j = \sum_{j'=0}^{d} x_{j'} \int_{s=\alpha}^{1} \varphi_{j'}(s)\varphi_j(s-\alpha)ds + \sum_{j'=0}^{d} x_{j'} \int_{s=0}^{\alpha} \varphi_{j'}(s)\varphi_j(s+1-\alpha)ds.$$

We denote by P the polynomial of degree less or equal than d such that $P(\alpha_j) = x_j$ for all $j = 0, ..., d$. Then we obtain

$$\begin{aligned} \lambda\omega_j P(\alpha_j) &= \sum_{j'=0}^{d} P(\alpha_{j'}) \int_{s=\alpha}^{1} \varphi_{j'}(s)\varphi_j(s-\alpha)ds \\ &\quad + \sum_{j'=0}^{d} P(\alpha_{j'}) \int_{s=0}^{\alpha} \varphi_{j'}(s)\varphi_j(s+1-\alpha)ds. \end{aligned} \tag{5.3}$$

We sum over j :

$$\begin{aligned} \lambda\sum_{j=0}^{d} \omega_j P(\alpha_j) &= \sum_{j'=0}^{d} P(\alpha_{j'}) \int_{s=\alpha}^{1} \varphi_{j'}(s)\sum_{j=0}^{d}\varphi_j(s-\alpha)ds \\ &\quad + \sum_{j'=0}^{d} P(\alpha_{j'}) \int_{s=0}^{\alpha} \varphi_{j'}(s)\sum_{j=0}^{d}\varphi_j(s+1-\alpha)ds \end{aligned}$$

and we get, by (5.1) and Gauss quadrature formula,

$$\lambda\int_{s=0}^{1} P(s)ds = \sum_{j'=0}^{d} P(\alpha_{j'}) \int_{s=\alpha}^{1} \varphi_{j'}(s)ds + \sum_{j'=0}^{d} P(\alpha_{j'}) \int_{s=0}^{\alpha} \varphi_{j'}(s)ds$$

then we use (5.2) and Gauss quadrature formula and we obtain finally

$$\lambda\int_{s=0}^{1} P(s)ds = \int_{s=0}^{1} P(s)ds. \tag{5.4}$$

We conclude that if $\lambda \neq 1$ then $\int_{s=0}^{1} P(s)ds = 0$.

We multiply (5.3) by $P(\alpha_j)$:

$$\lambda\omega_j P^2(\alpha_j) = \sum_{j'=0}^{d} P(\alpha_{j'})P(\alpha_j)\int_{s=\alpha}^{1}\varphi_{j'}(s)\varphi_j(s-\alpha)ds + \sum_{j'=0}^{d} P(\alpha_{j'})P(\alpha_j)\int_{s=0}^{\alpha}\varphi_{j'}(s)\varphi_j(s+1-\alpha)ds.$$

We sum over j :

$$\lambda\sum_{j=0}^{d}\omega_j P^2(\alpha_j) = \int_{s=\alpha}^{1}\left(\sum_{j'=0}^{d} P(\alpha_{j'})\varphi_{j'}(s)\right)\left(\sum_{j=0}^{d} P(\alpha_j)\varphi_j(s-\alpha)\right)ds + \int_{s=0}^{\alpha}\left(\sum_{j'=0}^{d} P(\alpha_{j'})\varphi_{j'}(s)\right)\left(\sum_{j=0}^{d} P(\alpha_j)\varphi_j(s+1-\alpha)\right)ds$$

and we get, since $\deg(P^2) \leq 2d \leq 2d+1$ and Gauss quadrature formula is still valid :

$$\lambda\int_0^1 P(s)^2 ds = \int_0^1 P(s)(1_{[\alpha,1[}(s)P(s-\alpha) + 1_{[0,\alpha]}(s)P(s+1-\alpha))ds. \tag{5.5}$$

By Cauchy-Schwarz inequality, we have

$$\left(\int_0^1 P(s)(1_{[\alpha,1[}(s)P(s-\alpha) + 1_{[0,\alpha]}(s)P(s+1-\alpha))ds\right)^2 \leq \int_0^1 P(s)^2 ds \cdot \int_0^1 (1_{[\alpha,1[}(s)P(s-\alpha) + 1_{[0,\alpha]}(s)P(s+1-\alpha))^2 ds.$$

The last term can be simplified :

$$\int_0^1 (1_{[\alpha,1[}(s)P(s-\alpha) + 1_{[0,\alpha]}(s)P(s+1-\alpha))^2 ds = \int_0^{1-\alpha} P(s)^2 ds + \int_{1-\alpha}^1 P(s)^2 ds = \int_0^1 P(s)^2 ds$$

then we obtain :

$$\left|\int_0^1 P(s)(1_{[\alpha,1[}(s)P(s-\alpha) + 1_{[0,\alpha]}(s)P(s+1-\alpha))ds\right| \leq \int_0^1 P(s)^2 ds. \tag{5.6}$$

The relations (5.5) and (5.6) lead to

$$|\lambda|\int_0^1 P(s)^2 ds \leq \int_0^1 P(s)^2 ds. \tag{5.7}$$

We have equality in (5.6) if and only if the functions $s \mapsto P(s)$ and $s \mapsto 1_{[\alpha,1[}(s)P(s-\alpha) + 1_{[0,\alpha]}(s)P(s+1-\alpha)$ are proportional *i.e.* there exists $(\mu_1,\mu_2) \neq (0,0)$ such that

$$\mu_1 P(s) = \mu_2(1_{[\alpha,1[}(s)P(s-\alpha) + 1_{[0,\alpha]}(s)P(s+1-\alpha)).$$

It is clear that a such relation is not possible if P is of degree 1 (we suppose $0 < \alpha < 1$) and if P has a degree greater than 1, we can differentiate the relation and the relation still remain the same for the derivatives which will be of degree 1 at a moment. Also P is necessarily constant.

If $|\lambda| = 1$ we have equality in (5.7) and then, by the previous remark, P is constant. Then, if $|\lambda| = 1$ and $\lambda \neq 1$, the relation (5.4) implies $P = 0$ which is not possible.

We consider the two subspaces $\mathcal{V} = \{P \in \mathbb{C}_d[X] \mid P \text{ is constant}\}$ and $\mathcal{W} = \{P \in \mathbb{C}_d[X] \mid \int_0^1 P(x)dx = 0\}$. These two subspaces are in direct sum associated to the decomposition $P(X) = \int_0^1 P(x)dx + \left(P(X) - \int_0^1 P(x)dx\right)$. These subspaces are moreover stable by D_0.

In fact, if $P \in \mathcal{V}$ we can assume that $P \equiv 1$. The j^{th}-component of $D_0 \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ reads

$$\frac{1}{\omega_j}\sum_{j'=0}^{d}\int_{s=\alpha}^{1}\varphi_{j'}(s)\varphi_j(s-\alpha)ds + \frac{1}{\omega_j}\sum_{j'=0}^{d}\int_{s=0}^{\alpha}\varphi_{j'}(s)\varphi_j(s+1-\alpha)ds = \frac{1}{\omega_j}\int_0^1 \varphi_j(s)ds = 1.$$

If $P \in \mathcal{W}$, we have $\int_0^1 P = \sum_{i=0}^{d}\omega_i P(\alpha_i) = 0$ then $\int_0^1 D_0 P$ reads

$$\begin{aligned}
& \sum_{j=0}^{d}\left(\sum_{j'=0}^{d}P(\alpha_{j'})\int_{s=\alpha}^{1}\varphi_{j'}(s)\varphi_j(s-\alpha)ds + \sum_{j'=0}^{d}P(\alpha_{j'})\int_{s=0}^{\alpha}\varphi_{j'}(s)\varphi_j(s+1-\alpha)ds\right) \\
= \; & \sum_{j'=0}^{d}P(\alpha_{j'})\int_{s=\alpha}^{1}\varphi_{j'}(s)\sum_{j=0}^{d}\varphi_j(s-\alpha)ds + \sum_{j'=0}^{d}P(\alpha_{j'})\int_{s=0}^{\alpha}\varphi_{j'}(s)\sum_{j=0}^{d}\varphi_j(s+1-\alpha)ds \\
= \; & \sum_{j'=0}^{d}P(\alpha_{j'})\int_{s=0}^{1}\varphi_{j'}(s)ds \\
= \; & \sum_{j'=0}^{d}\omega_{j'}P(\alpha_{j'}) \\
= \; & 0.
\end{aligned}$$

For summary, the matrix D_0 admits the eigenvalue $\lambda = 1$ associated to the eigenspace $\mathcal{V}$ of dimension 1 and others eigenvalues of modulus strictly less than 1 associated to the space $\mathcal{W}$ of dimension d.

The base vector e_i corresponds to the polynomial P_i defined by $P_i(\alpha_j) = \delta_{ij}$. The projection of P_i in subspace $\mathcal{V}$ gives $\int P_i = \sum_j P_i(\alpha_j)\omega_j = \omega_i$. So, we have

$$\lim_{n\to+\infty} D_0^n e_i = \omega_i \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

which completes the proof. □

We then consider the other matrices $D_1, \ldots, D_{N-1}$:

Proposition 5.3. For all $m = 1 \ldots N-1$, the matrix $D_m = A_0 + A_{-1}e^{\frac{2i\pi m}{N}}$ verifies

$$D_m^n \underset{n\to+\infty}{\longrightarrow} 0$$

Proof. In this proof, we will use as above the following relations :

$$\sum_{j=0}^{d} \varphi_j(x) = 1 \qquad \text{for all } 0 \leq x \leq 1 \tag{5.8}$$

$$\int_0^1 \varphi_j(s)ds = \omega_j. \tag{5.9}$$

Let λ be an eigenvalue of the matrice D_m. There exists $x = (x_0, ..., x_d)$ such that $D_m x = \lambda x$:

$$\lambda \omega_j x_j = e^{\frac{2i\pi m}{N}} \sum_{j'=0}^{d} x_{j'} \int_{s=\alpha}^{1} \varphi_{j'}(s)\varphi_j(s-\alpha)ds + \sum_{j'=0}^{d} x_{j'} \int_{s=0}^{\alpha} \varphi_{j'}(s)\varphi_j(s+1-\alpha)ds.$$

We denote by P the polynomial of degree less or equal than d such that $P(\alpha_j) = x_j$ for all $j = 0, ..., d$. Then :

$$\begin{aligned} \lambda \omega_j P(\alpha_j) &= e^{\frac{2i\pi m}{N}} \sum_{j'=0}^{d} P(\alpha_{j'}) \int_{s=\alpha}^{1} \varphi_{j'}(s)\varphi_j(s-\alpha)ds \\ &\quad + \sum_{j'=0}^{d} P(\alpha_{j'}) \int_{s=0}^{\alpha} \varphi_{j'}(s)\varphi_j(s+1-\alpha)ds. \end{aligned} \tag{5.10}$$

We sum over j :

$$\begin{aligned} \lambda \sum_{j=0}^{d} \omega_j P(\alpha_j) &= e^{\frac{2i\pi m}{N}} \sum_{j'=0}^{d} P(\alpha_{j'}) \int_{s=\alpha}^{1} \varphi_{j'}(s) \sum_{j=0}^{d} \varphi_j(s-\alpha)ds \\ &\quad + \sum_{j'=0}^{d} P(\alpha_{j'}) \int_{s=0}^{\alpha} \varphi_{j'}(s) \sum_{j=0}^{d} \varphi_j(s+1-\alpha)ds \end{aligned}$$

and we get, by (5.8) and Gauss quadrature formula,

$$\lambda \int_{s=0}^{1} P(s)ds = e^{\frac{2i\pi m}{N}} \sum_{j'=0}^{d} P(\alpha_{j'}) \int_{s=\alpha}^{1} \varphi_{j'}(s)ds + \sum_{j'=0}^{d} P(\alpha_{j'}) \int_{s=0}^{\alpha} \varphi_{j'}(s)ds$$

then we use (5.9) and Gauss quadrature formula and we obtain finally

$$(\lambda - 1) \int_{s=0}^{1} P(s)ds = (e^{\frac{2i\pi m}{N}} - 1) \int_{s=\alpha}^{1} P(s)ds. \tag{5.11}$$

We multiply (5.10) by $P(\alpha_j)$:

$$\begin{aligned} \lambda \omega_j P^2(\alpha_j) &= e^{\frac{2i\pi m}{N}} \sum_{j'=0}^{d} P(\alpha_{j'})P(\alpha_j) \int_{s=\alpha}^{1} \varphi_{j'}(s)\varphi_j(s-\alpha)ds \\ &\quad + \sum_{j'=0}^{d} P(\alpha_{j'})P(\alpha_j) \int_{s=0}^{\alpha} \varphi_{j'}(s)\varphi_j(s+1-\alpha)ds. \end{aligned}$$

We sum over j :

$$\lambda \sum_{j=0}^{d} \omega_j P^2(\alpha_j) = e^{\frac{2i\pi m}{N}} \int_{s=\alpha}^{1} \left(\sum_{j'=0}^{d} P(\alpha_{j'}) \varphi_{j'}(s) \right) \left(\sum_{j=0}^{d} P(\alpha_j) \varphi_j(s-\alpha) \right) ds$$
$$+ \int_{s=0}^{\alpha} \left(\sum_{j'=0}^{d} P(\alpha_{j'}) \varphi_{j'}(s) \right) \left(\sum_{j=0}^{d} P(\alpha_j) \varphi_j(s+1-\alpha) \right) ds$$

and we get, since $\deg(P^2) \leq 2d \leq 2d+1$ and Gauss quadrature formula is still valid :

$$\lambda \int_0^1 P(s)^2 ds = \int_0^1 P(s)(1_{[\alpha,1[}(s) e^{\frac{2i\pi m}{N}} P(s-\alpha) + 1_{[0,\alpha]}(s) P(s+1-\alpha)) ds. \tag{5.12}$$

By Cauchy-Schwarz inequality, we have

$$\left| \int_0^1 P(s)(1_{[\alpha,1[}(s) e^{\frac{2i\pi m}{N}} P(s-\alpha) + 1_{[0,\alpha]}(s) P(s+1-\alpha)) ds \right|^2 \leq$$
$$\int_0^1 P(s)^2 ds \cdot \int_0^1 \left| 1_{[\alpha,1[}(s) e^{\frac{2i\pi m}{N}} P(s-\alpha) + 1_{[0,\alpha]}(s) P(s+1-\alpha) \right|^2 ds.$$

The last term can be simplified, since the functions have distinct supports :

$$\int_0^1 \left| 1_{[\alpha,1[}(s) e^{\frac{2i\pi m}{N}} P(s-\alpha) + 1_{[0,\alpha]}(s) P(s+1-\alpha) \right|^2 ds$$
$$= \int_0^1 \left| 1_{[\alpha,1[}(s) e^{\frac{2i\pi m}{N}} P(s-\alpha) \right|^2 + \int_0^1 \left| 1_{[0,\alpha]}(s) P(s+1-\alpha) \right|^2 ds = \int_0^1 P(s)^2 ds$$

then we obtain :

$$\left| \int_0^1 P(s)(1_{[\alpha,1[}(s) e^{\frac{2i\pi m}{N}} P(s-\alpha) + 1_{[0,\alpha]}(s) P(s+1-\alpha)) ds \right| \leq \int_0^1 P(s)^2 ds \tag{5.13}$$

and the relations (5.12) and (5.13) lead to

$$|\lambda| \int_0^1 P(s)^2 ds \leq \int_0^1 P(s)^2 ds. \tag{5.14}$$

We have equality in (5.13) if and only if the functions $s \mapsto P(s)$ and $s \mapsto 1_{[\alpha,1[}(s) e^{\frac{2i\pi m}{N}} P(s-\alpha) + 1_{[0,\alpha]}(s) P(s+1-\alpha)$ are proportional *i.e.* there exists $(\mu_1, \mu_2) \neq (0,0)$ such that

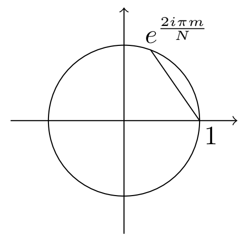
$$\mu_1 P(s) = \mu_2 (1_{[\alpha,1[}(s) e^{\frac{2i\pi m}{N}} P(s-\alpha) + 1_{[0,\alpha]}(s) P(s+1-\alpha)).$$

It's clear that a such relation is not possible if P is of degree 1 (we suppose $0 < \alpha < 1$) and if P has a degree greater than 1, we can derivate the relation and the relation still remain the same for the derivatives which will be of degree 1 at a moment. Also P is necessarily constant.

If $|\lambda| = 1$ we have equality in (5.14) and then, by the previous remark, P is constant. In this case, we obtain, using (5.11) :

$$\lambda = (1-\alpha) e^{\frac{2i\pi m}{N}} + \alpha$$

then we have proved the fact that λ belongs to the segment between 1 and $e^{\frac{2i\pi m}{N}}$ and finally the condition $0 < \alpha < 1$ implies that $|\lambda| < 1$.



Finally, all the eigenvalues have a modulus strictly less than 1 which completes the proof. □

Using the two previous propositions, we then can establish :

Corollary 5.4. We have the convergence property :

$$\mathcal{S}^n \underset{n\to+\infty}{\longrightarrow} UD^\infty U^\star$$

where

$$D^\infty = \begin{pmatrix} G & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}.$$

6. End of the proof

By Proposition 4.6, we have

$$\mathbf{e}^n = \Delta t \sum_{\ell=0}^{n-1} \mathcal{S}^\ell \mathbf{g}^{n-1-\ell},$$

which leads to

$$\|\mathbf{e}^n\|_2 \leq \Delta t \underbrace{\sum_{\ell=0}^{n-1} \|\mathcal{S}^\ell - UD^\infty U^\star\|_2 \|\mathbf{g}^{n-1-\ell}\|_2}_{(1)} + \Delta t \underbrace{\left\| \sum_{\ell=0}^{n-1} UD^\infty U^\star \mathbf{g}^{n-1-\ell} \right\|_2}_{(2)}.$$

<u>First term</u> : we note

$$\rho_d = max(|\lambda| \mid \lambda \text{ eigenvalue of } \mathcal{S} \; and \; |\lambda| < 1)$$

then we have the majoration

$$\|\mathcal{S}^\ell - UD^\infty U^\star\|_2 \leq \rho_d^\ell.$$

We have

$$\|\mathbf{g}^{n-1-\ell}\|_2 \leq C_d \frac{\Delta x^{d+1}}{\Delta t}$$

then

$$\sum_{\ell=0}^{n-1} \|\mathcal{S}^\ell - UD^\infty U^\star\|_2 \|\mathbf{g}^{n-1-\ell}\|_2 \leq \frac{1-\rho_d^n}{1-\rho_d} C_d \frac{\Delta x^{d+1}}{\Delta t} \leq C_1 \frac{\Delta x^{d+1}}{\Delta t}.$$

Second term : we show, by calculation, that the vector $UD^{\infty}U^{\star}\mathbf{g}^{\ell}$ is equal to

$$\left(\sum_{j=0}^{d}\omega_j\sum_{i=0}^{N-1}g_{ij}^{\ell}\right)\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}\in\mathbb{R}^{(d+1)N}$$

and we have, by Proposition 4.3 :

$$\begin{aligned}\sum_{i=0}^{N-1}\sum_{j=0}^{d}\omega_j\sum_{\ell=0}^{n-1}g_{ij}^{\ell} &= \sum_{\ell=0}^{n-1}\sum_{k=0}^{2d+1}\sum_{j=0}^{d}\omega_j E_j^k\frac{\Delta x^k}{\Delta t}\left(\sum_{i=0}^{N-1}\partial_x^k f(t^\ell,x_{ij})\right)+\mathcal{O}\left(\frac{\Delta x^{2d+2}}{\Delta t}\right)\\ &= \sum_{\ell=0}^{n-1}\sum_{k=0}^{2d+1}\sum_{j=0}^{d}\omega_j E_j^k\frac{\Delta x^k}{\Delta t}\left(\sum_{i=0}^{N-1}\partial_x^k f^0(x_{ij}-\ell\Delta t)\right)+\mathcal{O}\left(\frac{\Delta x^{2d+2}}{\Delta t}\right).\end{aligned}$$

The Euler-MacLaurin theorem states that if $(m,n)\in\mathbb{Z}^2$, $m<n$, $k\in\mathbb{N}^*$ and $f:[m,n]\to\mathbb{C}$ a $\mathcal{C}^r([m,n])$ function then we have :

$$\frac{f(m)}{2}+f(m+1)+\ldots+f(n-1)+\frac{f(n)}{2}=\int_m^n f(t)dt+\sum_{k=2}^{r}\frac{b_k}{k!}(f^{(k-1)}(n)-f^{(k-1)}(m))+R_r$$

with

$$R_r=\frac{(-1)^{r+1}}{r!}\int_m^n\tilde{B}_r(t)f^{(r)}(t)dt$$

where b_n are the Bernoulli numbers and $\tilde{B}_n$ the Bernoulli polynomials. We apply the Euler-MacLaurin theorem to the function

$$i\mapsto\partial_x^k f^0(x_{ij}-\ell\Delta t)=\partial_x^k f^0((i+\alpha_j)\Delta x-\ell\Delta t)$$

with $r=2d+2-k$:

$$\begin{aligned}\sum_{i=0}^{N-1}\partial_x^k f^0((i+\alpha_j)\Delta x-\ell\Delta t) &= \int_{t=0}^{N}\partial_x^k f^0((t+\alpha_j)\Delta x-\ell\Delta t)dt+R_{2d+2-k}^{j,\ell}\\ &= \frac{1}{\Delta x}\int_0^1\partial_x^k f^0(x)dx+R_{2d+2-k}^{j,\ell}\end{aligned}$$

where

$$R_{2d+2-k}^{j,\ell}=\frac{(-1)^{2d+3-k}}{(2d+2-k)!}\int_0^N\tilde{B}_{2d+2-k}(t)\partial_x^k f^{0\,(2d+2-k)}((t+\alpha_j)\Delta x-\ell\Delta t)dt.$$

Then we have

$$\begin{aligned}\sum_{i=0}^{N-1}\sum_{j=0}^{d}\omega_j\sum_{\ell=0}^{n-1}g_{ij}^{\ell} &= \sum_{\ell=0}^{n-1}\sum_{k=0}^{2d+1}\frac{\Delta x^k}{\Delta t}\left(\frac{1}{\Delta x}\int_0^1\partial_x^k f^0(x)dx\right)\left(\sum_{j=0}^{d}\omega_j E_j^k\right)\\ &\quad+\sum_{\ell=0}^{n-1}\sum_{k=0}^{2d+1}\sum_{j=0}^{d}\omega_j E_j^k\frac{\Delta x^k}{\Delta t}R_{2d+2-k}^{j,\ell}+\mathcal{O}\left(\frac{\Delta x^{2d+2}}{\Delta t}\right)\end{aligned}$$

We use Proposition 4.5 in order to conclude that

$$\sum_{\ell=0}^{n-1}\sum_{k=0}^{2d+1}\frac{\Delta x^k}{\Delta t}\left(\frac{1}{\Delta x}\int_0^1\partial_x^k f^0(x)dx\right)\left(\sum_{j=0}^{d}\omega_j E_j^k\right)=0$$

and $R^{j,\ell}_{2d+2-k} = \mathcal{O}(\Delta x^{2d+2-k})$ leads to

$$\left\|\sum_{\ell=0}^{n-1} UD^\infty U^\star \mathbf{g}^{n-1-\ell}\right\|_2 \leq nC_2 \frac{\Delta x^{2d+2}}{\Delta t}.$$

Conclusion. The two previous paragraphs show the theorem :

$$\|\mathbf{e}^n\|_2 \leq C_1 \Delta x^{d+1} + nC_2 \Delta x^{2d+2}.$$

7. Symbolic and numerical results

7.1. **Symbolic results.** The scheme is given by $F^{n+1}_j = A_{-1}F^n_{j-1} + A_0 F^n_j$ (see the beginning of Section 5). We consider

$$\hat{F}^n_k = \frac{1}{N}\sum_{j=0}^{N-1} F^n_j e^{\frac{-2i\pi jk}{N}}.$$

By taking the initial condition :

$$f(0,x) = e^{2i\pi\ell x}$$

we obtain :

$$(\hat{F}^0_k)_i = e^{\frac{2i\pi k\alpha_i}{N}}.$$

The amplification matrix of the scheme $\widehat{F}^{n+1}_k = \widehat{\mathcal{A}}_k \widehat{F}^n_k$ reads

$$\widehat{\mathcal{A}}_k = A_0 + A_{-1}e^{\frac{-2ik\pi}{N}}.$$

We denote by $\lambda_{0,k},...,\lambda_{d,k}$ and $V_{0,k},...,V_{d,k}$ the eigenvalues and eigenvectors of $\widehat{\mathcal{A}}_k$. The solution of the scheme is given by :

$$\widehat{F}^n_k = (\widehat{\mathcal{A}}_k)^n \widehat{F}^0_k.$$

We choose the eigenvectors $V_{0,k},...,V_{d,k}$ such that $\widehat{F}^0_k = \sum_{j=0}^d V_{j,k}(\alpha)$ then we have

$$\widehat{F}^n_k = \sum_{j=0}^d \lambda^n_{j,k} V_{j,k}(\alpha).$$

The error in Fourier space then reads :

$$\sum_{j=0}^d (\lambda^n_{j,k} - e^{-2i\pi k n\alpha\Delta x}) V_{j,k}(\alpha).$$

We restrict the study to the case $d = 1$ and note $\omega := 2\pi k$. We obtain the eigenvalues and eigenvectors by using Maple :

$$\begin{aligned}
\lambda_{0,\omega} &= 1 - \alpha i\omega\Delta x + \frac{1}{2}(\alpha i\omega\Delta x)^2 - \frac{1}{6}(\alpha i\omega\Delta x)^3 + \frac{\alpha(4\alpha^3-2\alpha^2+2\alpha-1)}{72}(i\omega\Delta x)^4 + \mathcal{O}(\Delta x^5),\\
\lambda_{1,\omega} &= 6\alpha^2 - 6\alpha + 1 + \mathcal{O}(\Delta x),\\
V_{0,\omega} &= \begin{pmatrix} 1 + \frac{3-\sqrt{3}}{6} i\omega\Delta x + \mathcal{O}(\Delta x^2) \\ 1 + \frac{3+\sqrt{3}}{6} i\omega\Delta x + \mathcal{O}(\Delta x^2)\end{pmatrix},\\
V_{1,\omega} &= \begin{pmatrix} -\frac{\sqrt{3}(2\alpha-1)}{36}(i\omega\Delta x)^2 - \frac{-4\sqrt{3}\alpha^2+(10\sqrt{3}-6)\alpha+(3-5\sqrt{3})}{216}(i\omega\Delta x)^3 + \mathcal{O}(\Delta x^4) \\ \frac{\sqrt{3}(2\alpha-1)}{36}(i\omega\Delta x)^2 - \frac{4\sqrt{3}\alpha^2+(-10\sqrt{3}-6)\alpha+(3+5\sqrt{3})}{216}(i\omega\Delta x)^3 + \mathcal{O}(\Delta x^4)\end{pmatrix}.
\end{aligned}$$

The computation of the error in Fourier space gives

$$\begin{pmatrix} -\frac{\sqrt{3}(2\alpha-1)\mathcal{T}_1(n,\alpha)}{36}(i\omega\Delta x)^2 - \frac{\mathcal{T}_2(n,\alpha)}{216(6\alpha^2-6\alpha+1)}(i\omega\Delta x)^3 + \mathcal{O}(\Delta x^4) \\ \frac{\sqrt{3}(2\alpha-1)\mathcal{T}_1(n,\alpha)}{36}(i\omega\Delta x)^2 - \frac{\mathcal{T}_3(n,\alpha)}{216(6\alpha^2-6\alpha+1)}(i\omega\Delta x)^3 + \mathcal{O}(\Delta x^4) \end{pmatrix}$$

where

$$\begin{aligned} \mathcal{T}_1(n,\alpha) &= (6\alpha^2-6\alpha+1)^n - 1, \\ \mathcal{T}_2(n,\alpha) &= n\sqrt{3}(72\alpha^4-108\alpha^3+48\alpha^2-6\alpha) \\ &\quad + n\sqrt{3}(6\alpha^2-6\alpha+1)^n(24\alpha^4-84\alpha^3+72\alpha^2-18\alpha) \\ &\quad + [(6\alpha^2-6\alpha+1)^n-1][-24\sqrt{3}\alpha^4+(-36+84\sqrt{3})\alpha^3 \\ &\quad + (54-74\sqrt{3})\alpha^2+(-24+40\sqrt{3})\alpha+3-5\sqrt{3}], \end{aligned}$$

$$\begin{aligned} \mathcal{T}_3(n,\alpha) &= -n\sqrt{3}(72\alpha^4-108\alpha^3+48\alpha^2-6\alpha) \\ &\quad - n\sqrt{3}(6\alpha^2-6\alpha+1)^n(24\alpha^4-84\alpha^3+72\alpha^2-18\alpha) \\ &\quad + [(6\alpha^2-6\alpha+1)^n-1][24\sqrt{3}\alpha^4+(-36-84\sqrt{3})\alpha^3 \\ &\quad + (54+94\sqrt{3})\alpha^2+(-24-40\sqrt{3})\alpha+3+5\sqrt{3}]. \end{aligned}$$

We validate, in this case, the estimation of the error :

$$\|\mathbf{e}^n\|_2 \leq C_1\Delta x^{d+1} + nC_2\Delta x^{2d+2}.$$

7.2. **Numerical results.** We process a numerical study of the convergence of the scheme for $d = 1$ and $d = 2$ (Fig. 1). We see that in both cases, the error is of order $d+1$ when we iterate the scheme only one time. In fact, for a low number of iterations, the dominant term in the error bound

$$\|\mathbf{e}^n\|_2 \leq C_1\Delta x^{d+1} + nC_2\Delta x^{2d+2}$$

is $C_1\Delta x^{d+1}$. When the number of iterations n increases, the dominant term of the error becomes $nC_2\Delta x^{2d+2}$ and then we show the emergence of a slope of order $2d+2$ for large values of Δx.

8. Conclusion

We prove a superconvergence property for the Semi-Lagrangian Discontinuous Galerkin scheme. An adaptation of this proof when α goes to 0 in order to show this property for the scheme studied in [6] could be a further work. Such a superconvergence property in the case of the Vlasov-Poisson equation is a completely open problem.

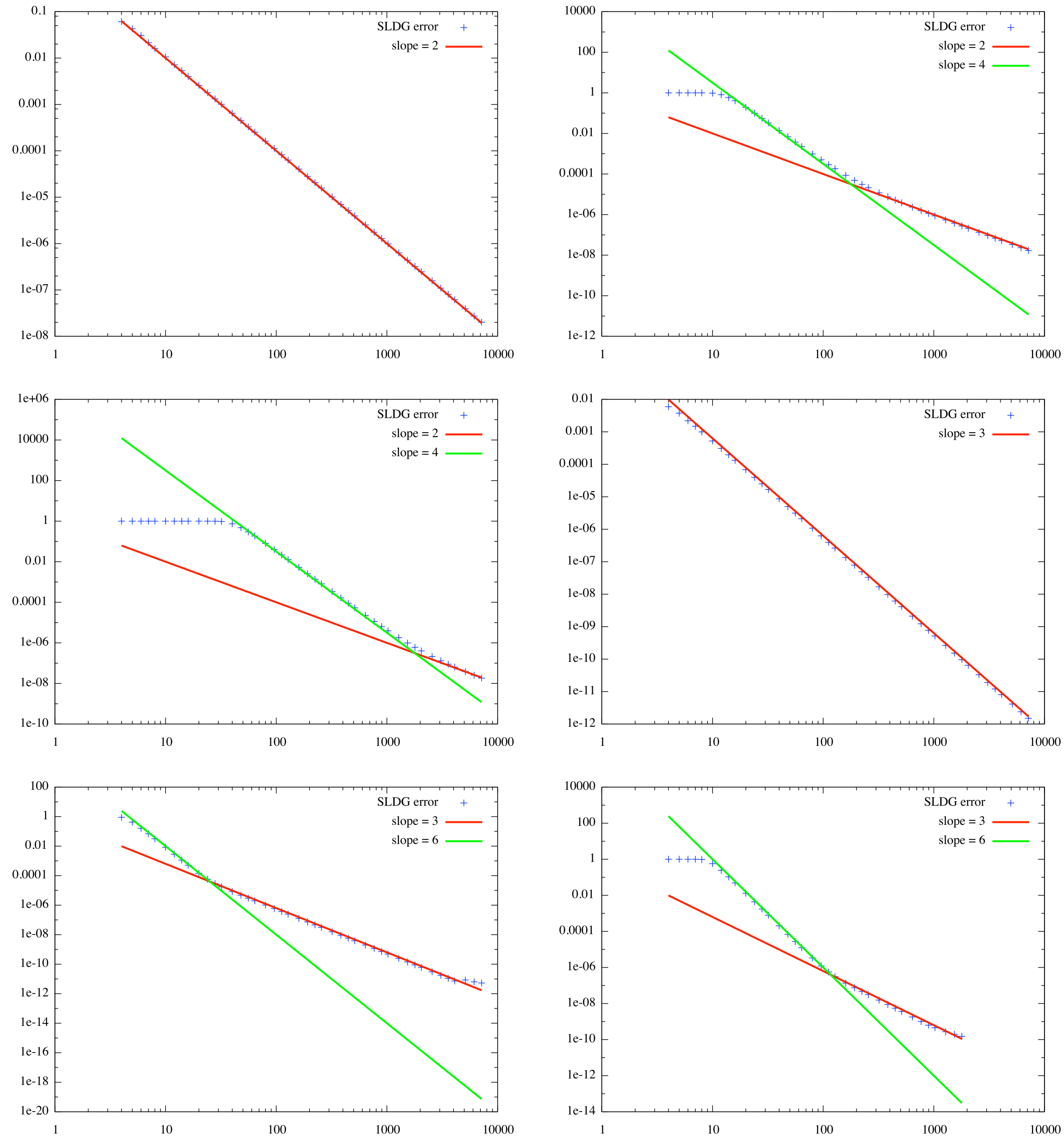


FIGURE 1. SLGD error convergence for $d = 1$ and 1 iteration (top left), 10^4 iterations (top right) and 10^6 iterations (middle left). SLGD error for $d = 2$ and 1 iteration (middle right), 10^4 iterations (bottom left) and 10^6 iterations (bottom right). We have used $f_0(x) = \sin(2\pi x)$ and $a = 1$.

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